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Corresponding Author: Dr. S Y Novak,

Corresponding Author's Institution:

First Author: S Y Novak

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Abstract: We evaluate the accuracy of approximation to the distribution of the length of the longest head run in a Markov chain with a discrete state space. An estimate of the accuracy of approximation in terms of the total variation distance is established for the first time.

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Response to the reviewer comments

1. I've replaced "length of the longest head run" with "longest run" as suggested by the reviewer.
2. I've added " $\{\xi_k=1\}$ represents failure of the k th component" as suggested by the reviewer.
3. The text on page 2 states " $N_n(k)$ of head runs with lengths $\geq k$ ".
4. Statistics & Probability Letters is among journals cited in the manuscript.

20.06.2017

On the length of the longest head run

S.Y.Novak

MDX University, School of Science & Technology,
The Burroughs, London NW44BT, UK

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Abstract

We evaluate the accuracy of approximation to the distribution of the length of the longest head run in a Markov chain with a discrete state space. An estimate of the accuracy of approximation in terms of the total variation distance is established for the first time.

Key words: longest head run, extremes in samples of random size.

AMS Subject Classification: 60E15, 60G70.

1 Introduction

Let $\{\xi_i, i \geq 1\}$ be a sequence of random 0's and 1's (i.e., “tails” and “heads”). Then

$$L_n = \max\{k: \xi_{i+1} = \dots = \xi_{i+k} = 1 \quad (\exists i \leq n-k)\} \quad (1)$$

is the length of the longest head run (LLHR) among ξ_1, \dots, ξ_n .

Statistic L_n has applications in biology, reliability theory, finance, and nonparametric statistics (see, e.g., [1, 2, 3, 17]). In particular, the reliability of a consecutive k -out-of- n system with n components can be expressed via $\mathbb{P}(L_n < k)$, where the event $\{\xi_k = 1\}$ represents failure of the k th component: the system fails if and only if k consecutive components fail [1, 4, 6, 13].

The study of the distribution of LLHR has a long history. Apparently, the task was first formulated by de Moivre [10], Problem LXXIV. Renewed interest to the topic arose in connection with the Erdős–Rényi strong law of large numbers [5].

A limit theorem for LLHR in the case of independent Bernoulli $\mathbf{B}(p)$ trials was established by Goncharov [8]. The limiting distribution of LLHR was found in more general situations as well, see [1, 12, 14, 19] and references therein. In particular, a limit theorem for LLHR in a Markov chain with a finite state space \mathcal{X} where hitting a subset of \mathcal{X} is considered a “success” is given in [12]. An estimate of the rate of convergence and asymptotic expansions in the limit theorem for LLHR in a two-state Markov chain have been established in [13]. Concerning LIL-type results, see [16] and references therein.

An exact formula for $\mathbb{P}(L_n < k)$ in terms of combinatorial coefficients in the case of independent Bernoulli trials was found by Uspensky [18]. In the case of a two-state Markov chain Fe et al. [6] present an exact formula for $\mathbb{P}(L_n < k)$ in terms of a specially constructed matrix of transition probabilities, and establish the asymptotics of $\ln \mathbb{P}(L_n < k)$ as $n \rightarrow \infty$ if k is fixed (see also Theorem 2 in [13]).

Note that L_n can be represented as a sample maximum in a sample of random size ν_n , where ν_n is a certain renewal process (cf. [13, 14]). References concerning extremes in samples of random size can be found, e.g., in [7, 9, 16].

It is known that the accuracy of approximation to the distribution of LLHR in terms of the uniform distance is $n^{-1} \ln n$ [13]. The result has been generalised to the case of a Markov chain with a finite state space [14] as well as to the case of m -dependent random variables [15]. Asymptotic expansions in the limit theorem for LLHR in a two-state Markov chain [13] confirm that the rate $n^{-1} \ln n$ cannot be improved.

There is a simple relation between LLHR and the number $N_n(k)$ of head runs with lengths $\geq k$:

$$\{L_n < k\} = \{N_n(k) = 0\}.$$

Note that the estimates of the accuracy of approximation to the distribution of $N_n(k)$ have been established in terms of the total variation distance (see [1, 2, 16] and references therein). However, the problem of evaluating the accuracy of approximation to the distribution of LLHR in terms of the total variation distance remained open for a long while.

In this paper we derive an estimate of order $n^{-1} \ln n$ to the total variation distance between $\mathcal{L}(L_n)$ and the approximating distribution.

2 Results

Let $\{X_i, i \geq 1\}$ be a homogeneous Markov chain with a finite state space \mathcal{X} and transition probabilities $\|p_{ij}\|_{i,j \in \mathcal{X}}$. We denote by

$$\bar{\pi} = \|\pi_i\|_{i \in \mathcal{X}}$$

the stationary distribution of the chain.

Given a subset $A \subset \mathcal{X}$, let LLHR be defined by (1), where

$$\xi_i = \mathbb{I}\{X_i \in A\}$$

(hitting A is considered a “success”). We set

$$U = \|p_{ij}\|_{i,j \in A}, \quad \bar{\pi}_A = \|\pi_i\|_{i \in A},$$

and let

$$q(k) = \bar{\pi}_A U^{k-1} (E - U) \bar{1} \quad (k \geq 1),$$

where $\bar{1}$ is a vector of 1's and E is a unit diagonal matrix.

Let ζ_n, Z_n be random variables (r.v.s) with distribution functions (d.f.s)

$$\mathbb{P}(\zeta_n < k) = (1 - q(k))^{n-k}, \quad \mathbb{P}(Z_n < k) = \exp(-nq(k)) \quad (k \geq 1).$$

Recall the definition of the total variation distance between the distributions of r.v.s X and Y :

$$d_{TV}(X; Y) \equiv d_{TV}(\mathcal{L}(X); \mathcal{L}(Y)) = \sup_{A \in \mathcal{A}} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|,$$

where \mathcal{A} is a Borel σ -field.

The distribution of LLHR L_n can be well approximated by $\mathcal{L}(\zeta_n)$ or $\mathcal{L}(Z_n)$; the accuracy of such approximation in terms of the uniform distance is known to be of order $n^{-1} \ln n$. In Theorem 1 below we show that the result holds in terms of the stronger total variation distance.

Theorem 1 *Assume that*

(P0) *there is only one class C of essential states that consists of periodic subclasses C_1, \dots, C_d ;*

(P1) *$A \cap C_i \neq \emptyset$ ($1 \leq i \leq d$);*

- (P2) $0 < \lambda < 1$, where λ is the largest eigenvalue of matrix U ;
(P3) if $i \in C_\ell$ for some $\ell \in \{1, \dots, d\}$, then

$$|p_{ij}(m) - d\pi_j| \leq u_m, \quad H := \sum_{m \geq 1} u_m < \infty \quad (2)$$

if $j \in A \cap C_k$ and $k - m = \ell \pmod{d}$; if $i \notin C_1 \cup \dots \cup C_d$, then (2) holds for all $j \in A$;

(P4) $z_i > 0$ ($\forall i \in A$), where $\bar{z} = \|z_i\|_{i \in A}$ is the corresponding to λ right eigenvector of matrix U .

Then there exists a positive constant $C = C(\lambda, \bar{z}, \bar{\pi}_A)$ such that

$$d_{TV}(L_n; Z_n) \leq Cn^{-1} \ln n \quad (n \geq C). \quad (3)$$

The result holds if Z_n in (3) is replaced with ζ_n .

3 Proofs

Proof of Theorem 1 makes use of Theorem 2 from [14], which is presented below (note that the argument of Theorem 2 in [14] is valid for any fixed $d \in \mathbb{N}$). In the particular case of a stationary Markov chain the result of Theorem 2 is given by Theorem 2.1 in [12].

Theorem 2 Let $\{X_i, i \geq 1\}$ be a homogeneous Markov chain with a discrete state space \mathcal{X} , transition probabilities $\|p_{ij}\|_{i,j \in \mathcal{X}}$ and stationary distribution $\bar{\pi}$. Assume conditions P(0) – P(4). Then there exists a positive constant $c_\star = c_\star(\lambda, \bar{z}, \bar{\pi}_A)$ such that as $n > 2k \geq c_\star$,

$$\begin{aligned} & |\mathbb{P}(L_n < k) - \mathbb{P}(Z_n < k)| \\ & \leq c_\star \lambda^k + c_\star k \lambda^k \exp(-nq(k)(1 - c_\star k \lambda^k)). \end{aligned} \quad (4)$$

Taking into account the obvious inequality

$$|e^x - e^y| \leq |x - y| e^{\max\{x, y\}} \quad (x, y \in \mathbb{R}), \quad (5)$$

we notice that (4) holds true if Z_n is replaced with ζ_n .

In the case of independent observations inequalities of this kind with explicit constants are presented in [15, 11]. In the case of a two-state Markov chain with

$\alpha := p_{11} \in (0; 1)$, $\beta := p_{00} < 1$, a sharp bound of this kind is given in [13], Theorem 2: there exist constants $q \in (0; 1)$, $C < \infty$ such that

$$\sup_{k > C} \left| \mathbb{P}(L_n < k) - A(t_0)/t_0^{n+1} \right| \leq Cq^n \quad (6)$$

for particular t_0 and function $A(t)$ obeying $|A(t_0) - 1| \leq C_1 \gamma k \alpha^k$, $|t_0 - 1 - \gamma \alpha^k| \leq C_1 k (\gamma \alpha^k)^2$ for some $C_1 < \infty$, where $\gamma = (1-\alpha)(1-\beta)/\alpha(2-\alpha-\beta)$. In the case of independent Bernoulli $\mathbf{B}(\alpha)$ trials (6) holds with $q = \alpha$, $C = (2+\alpha-\alpha^2)/(1-\alpha)(1-\alpha^2)$.

By a well-known property of the total variation distance,

$$2d_{TV}(L_n; Z_n) = \sum_{k \geq 0} |\mathbb{P}(L_n = k) - \mathbb{P}(Z_n = k)|. \quad (7)$$

The idea of the proof is to split the sum in (7) into appropriate fragments and show that the desired estimate holds for each fragment.

Recall that $\bar{\pi}_A = \|\pi_i\|_{i \in A}$, and set

$$\begin{aligned} c_* &= \langle \bar{\pi}_A; \bar{z} \rangle (1-\lambda)/\lambda z^*, \quad c^* = \langle \bar{\pi}_A; \bar{z} \rangle (1-\lambda)/\lambda z_*, \\ \bar{z}_* &= \inf\{z_j : j \in A\}, \quad \bar{z}^* = \sup\{z_j : j \in A\}. \end{aligned}$$

Note that

$$0 < c_* \leq c^* < \infty.$$

It is easy to see that

$$c_* \lambda^k \leq q(k) \leq c^* \lambda^k \quad (8)$$

(cf. (8) in [14]). Let

$$k(n) = \log n - \log \ln n + \log(c_*/2).$$

Hereinafter \log is to the base $1/\lambda$, symbol c (with or without indexes) denotes positive constants.

Using (4) and (8), we check that

$$\begin{aligned} & \sum_{k \leq k(n)} |\mathbb{P}(L_n = k) - \mathbb{P}(Z_n = k)| \\ & \leq \mathbb{P}(L_n \leq k(n)) + \mathbb{P}(Z_n \leq k(n)) \leq c_1 n^{-1} \ln n. \end{aligned} \quad (9)$$

It remains to evaluate

$$\sum_{k>k(n)} |\mathbb{P}(L_n=k) - \mathbb{P}(Z_n=k)|.$$

According to (4) and (8), there exists a positive constant c_2 such that

$$|\mathbb{P}(L_n=k) - \mathbb{P}(Z_n=k)| \leq c_2 \lambda^k + c_2 k \lambda^k e^{-n\lambda^k c_*/2} \quad (10)$$

as $n > 2k \geq c_2$. Evidently,

$$\sum_{k>k(n)} \lambda^k \leq \lambda^{k(n)}/(1-\lambda) = 2n^{-1}(\ln n)/(1-\lambda)c_*. \quad (11)$$

Thus, it remains to evaluate $\sum_{k>k(n)} k \lambda^k e^{-n\lambda^k c_*/2}$.

Note that function $f(x) = xe^{-x}$ decreases in $[1; \infty)$. Clearly, $n\lambda^k c_*/2 \in [1; \ln n]$ as $k(n) < k < \log(nc_*/2)$. Therefore,

$$\begin{aligned} \sum_{k(n) < k < \log(nc_*/2)} k \lambda^k e^{-n\lambda^k c_*/2} &\leq n^{-1} \log(nc_*/2) \sum_{k(n) < k < \log(nc_*/2)} n \lambda^k e^{-n\lambda^k c_*/2} \\ &\leq n^{-1} \log(nc_*/2) \int_{k(n)}^{\log(nc_*/2)} n \lambda^x e^{-n\lambda^x c_*/2} dx \\ &\leq 2n^{-1} \log(nc_*/2) / \ln(1/\lambda) c_*. \end{aligned} \quad (12)$$

Since

$$\sum_{k \geq m} k \lambda^k \leq m \lambda^m / (1-\lambda)^2 \quad (m \geq 1),$$

we have

$$\sum_{k \geq \log(nc_*/2)} k \lambda^k e^{-n\lambda^k c_*/2} \leq \sum_{k \geq \log(nc_*/2)} k \lambda^k \leq 2 \lceil \log(nc_*/2) \rceil / nc_* (1-\lambda)^2, \quad (13)$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x .

Combining estimates (9) – (13), we derive (3). The proof is complete. \square

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